Chaos and Graphics

Wild knots

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Abstract

Wild knots, a class of knots which cannot be represented by polygonal paths in 3-dimensional space, are investigated as an embellishment to the Celtic style of ornamental knotwork. Wild knotwork designs are compared to fractal knotwork designs and the traditional technique of $N$-interlacement. It is shown that all three styles may co-exist in one design.

Keywords: Wild knot; Celtic knotwork; $N$-interlacement; Fractal; Art

1. Introduction

Celtic knotwork is an ornamental art style in which interlaced cords trace complex and well-structured designs to fill a given area. These designs consist of closed cords of finite length with an alternating weave, and it is a mark of the artist’s skill to create complex designs from as few cords as possible, ideally a single cord. See [1] for a brief summary of the key points of this style of artwork, and [2,3] for a more in-depth treatment.

Fig. 1(left) shows a simple plait in the Celtic style. Note that the design consists of a single cord and that the grid upon which it is based is rectangular with sides in the ratio 3:4; the number of cords in a simple plait will equal the greatest common divisor of the grid’s number of horizontal and vertical units.

In the more pragmatic terms of knot theory, a knot is a closed curve with no self-intersections that cannot be unknotted to produce a simple loop [4,5]. A link is a set of one or more knots with mutual entanglements; hence, a knot is a link with one component. Traditional knotwork designs are generally knots, although this is not always the case; in this paper, the term design shall be understood to mean a link that are equivalent to polygons are described as tame, and curves that are not are described as wild [5,7]. Similarly, if we define polygonal knots as those knots equivalent to piecewise-continuous polygons with a finite number of vertices and no self-intersections, then tame knots are those equivalent to polygonal knots and wild knots are those that are not (please see [5] for further details).

The simple plait of Fig. 1 (left) is a tame knot, as the cord’s path can be described by a polygon. However, Fig. 1 (right) is a wild knot even though its outer frame is based on the same design; this figure converges to a single closed curve as its resolution approaches infinity, but it is not equivalent to a polygon.

This wild knot can be recursively constructed using the set of generator curve segments shown in Fig. 2(left). The generator set has six exterior attachment points and six interior attachment points. To ensure a smooth transition between each generation, the curve ends at the exterior and interior attachment points must match in number, position and angle for the appropriate reduction factor (50% in this case).

Furthermore, intergenerational cord widths for each segment must be attenuated according to the following rules:

a) outer-to-outer: constant cord width (no attenuation),

b) outer-to-inner: attenuate towards interior end point,

c) inner-to-inner: attenuate towards both end points.

Example cord segments showing these rules are labeled a, b and c in Fig. 2(left). All figures in this paper were generated...
using director spline curves for the main cord paths, and variable offset curves [8] for the attenuated cord outlines.

Fig. 2 (right) shows the outer frame within which the wild knot is constructed. This is identical to the corresponding section of the tame knot, except that the wild version is distorted slightly to allow for the 3:4 aspect ratio during the 50% reduction for each generation. Note that the outer frame is also equivalent to the generator set, but with the exterior attachment points rounded and spliced to form continuous cords.

Fig. 3 (left) shows the first generation of wild knot construction. The generator set is reduced by 50% and its component cord segments grafted onto the corresponding segments of the outer frame; these then become the exposed cord ends awaiting further growth. With each generation there are always six internal attachment points waiting to be met, as the outer frame cord segments extend with diminishingly smaller steps towards the central attraction point.

To ensure that the final design forms a single continuous cord, it can be assumed that the exposed end points have been closed with a terminating “plug” as shown in Fig. 3 (right) at the ultimate level of recursion. The knot is therefore composed of the outer frame, \( r \) levels of the generator set, and the terminating plug. The resulting knot is tame if \( r \) is finite, but is wild in the limit as \( r \)
approaches infinity. Note that the plug is identical to the unused (central) part of the original 3:4 plait, hence every part of the plait is eventually incorporated into the final wild knot.

3. Tame fractal links

Fig. 4 shows two recursive designs constructed using similar principles, although in these cases the generator sets are themselves tame knots without attachment points, and the interaction between levels comes through interleavement of the tame cords at successive levels. These designs are fractal in the sense that the design is similar at all scales; it will look the same no matter how far the view is zoomed in.

Note that while Fig. 1 (right) consists of a single wild knot, the two designs shown in Fig. 4 consist of an infinite number of tame knots, one for each level. The tame fractal links do not require a special outer frame as all levels are identical, apart from rotation and scaling.

4. N-interlacement

N-interlacement is a traditional technique for adding complexity to knotwork designs by splitting cords into two or more parallel subcords, which are then subsequently rewoven to ensure an alternating weave [3]. Fig. 5 shows the basic 3:4 plait as a 2-interlaced design (left) and a 3-interlaced design (right).

N-Interlacement may be achieved simply by offsetting the main cord path using the attenuation rules above to give N parallel subcord paths, then offsetting these, again applying the attenuation rules, to give the final subcord outlines.

Note that N-interlacement results in N times the number of original cords; each level of interlacement removes the design one step further from the Celtic ideal of a single woven cord. However, this problem can be readily addressed by introducing breaks and crossovers into the design, as described in the next section.

5. Breaks and crossovers

A break consists of splitting two cords at the point where they cross and rejoining the cross-cords to inflect relative to either of the two possible axes of reflection; these will be the horizontal or vertical axes in the case of the rectangular grid.

Fig. 6 (left) shows a design on the 3:3 grid consisting of three cords—note that the greatest common divisor of
(3, 3) is 3. The middle figure shows this design reduced to two cords by adding a vertical break (bottom center) and the rightmost figure shows this design further reduced to a single cord by introducing a horizontal break (top right).

This simple operation is used by traditional knotwork designers to enhance the complexity of a design and manipulate its cord count. As J. Romilly Allen, one of the pioneers of Celtic knotwork analysis, states: “By continuing the process [of applying breaks] all the knots most commonly used in Celtic decorative art may be derived from a simple plait” [9]. Some skill is required in applying breaks to satisfy both the geometric constraints of the style (minimal number of cords) and the artist’s aesthetic preferences (complexity, symmetry, originality, and so on).

The opposite operation, which takes two adjacent cords running in parallel and introduces a crossing, may be called a crossover.

6. Putting it all together

Fig. 7 shows a hybrid design that incorporates the techniques described in the previous sections. The intergenerational reduction factor for this design is approximately 59%.

The main cord paths consist of a recursively repeated tame knot (sharp corners) interwoven with a wild knot (rounded clover shape). The main cord paths have been 2-interlaced, and the resulting paired subcords shaded light and dark for both the wild and tame components.

The total number of cords $C$ in this design, for recursion level $r$, is given by

$$C_r = 2 + 2r,$$

as the wild component contributes two cords, and there are two tame cords at each level of recursion.

This design incorporates typical Celtic motifs, but violates a key rule of the Celtic style; the wild component converges to a central point at which more than two cords cross as its limit approaches infinity. In addition, no attempt has been made to reduce the number of cords using break or crossover operations, not even within each layer of the tame fractal component.

Fig. 8 shows a more complex hybrid design which has been treated to more fully fit the Celtic ideal. The intergenerational reduction factor for this design is $33\frac{1}{8}\%$.

The main cross design, shown as a dark color, is a tame knot consisting of a single cord. The main cord path was broken in the middle of each of the upper three arms for artistic purposes, but remains unbroken in the lower arm to give a single path. The main cord path was then 2-interlaced, and another vertical break introduced in the lower arm subcord (center left) to reduce the final generator set for the cross to a single cord. Unlike the examples shown in Fig. 4, this knot’s fractal occurrence does not interweave between levels.

The light colored knot interwoven through the interstices of the main cross design is a wild knot that provides the connection between levels. The main cord path for the wild knot was broken vertically at the left side of the lower arm to ensure a single main path, and the 2-interlaced subcords crossed over at the same point in the outer frame (but not the generator set) to ensure that the two parallel subcords...
of the final design are spliced into a single wild cord. Again, terminating plugs are assumed to close exposed cord ends at the limit \( r \) of recursion to ensure a single wild knot.

The total number of cords \( C \) in this design, for recursion level \( r \), is therefore,

\[
C_r = 1 + \sum_{i=0}^{r} 4^i,
\]

where the single wild knot (light) contributes one cord, and the sigma component describes the single tame parent knot plus its four fractal children per generation (dark).

7. Conclusion

This paper demonstrates how the principles of wild knots and fractal links may be introduced as an embellishment to knotwork designs in the Celtic style. The construction of wild knot designs may be achieved by the recursive application of a simple generator set of pre-defined cord segments, provided that their end points satisfy certain geometric constraints.

These principles, together with the traditional knotwork technique of \( \mathcal{N} \)-interlacement, may be used to embellish a design and add complexity. Cord break and crossover operations may be applied to both the generator set and as a post-construction step to the overall design, in order to manipulate the design layout and cord number.

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References


